Multiple solutions for a class of nonlinear elliptic equations on the Sierpiński gasket

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Received December 2, 2003

Abstract This paper investigates a class of nonlinear elliptic equations on a fractal domain. We establish a strong Sobolev-type inequality which leads to the existence of multiple non-trivial solutions of \( \Delta u + c(x)u = f(x, u) \), with zero Dirichlet boundary conditions on the Sierpiński gasket. Our existence results do not require any growth conditions of \( f(x, u) \) in \( u \), in contrast to the classical theory of elliptic equations on smooth domains.

Keywords: Sierpiński gasket, energy form, Laplacian operator, eigenvalue problem, genus, weak solution.

DOI: 10.1360/02ys0366

1 Introduction

We are concerned with the existence of non-trivial solutions of a class of nonlinear elliptic equations

\[
\Delta u + c(x)u = f(x, u), \quad x \in K \setminus K_0, \\
u|_{K_0} = 0,
\]

where \( K \) is the Sierpiński gasket, \( K_0 \) is its boundary, and \( \Delta \) is the “Laplacian” operator on \( K \), defined below. The coefficient of linear term \( c(x) \) is non-negative with \( \int_K c(x) \mu > 0 \), where \( \mu \) denotes the restriction to \( K \) of normalized \( \frac{\log 2}{\log 3} \)-dimensional Hausdorff measure on \( K \), so that \( \mu(K) = 1 \), and \( f(x, t) \) is the nonlinear part and of class \( C^1 \) in \( t \) for all \( x \) and satisfies the following conditions:

(i) \( f \) is odd in \( t \), that is, \( f(x, -t) = -f(x, t) \) for all \( x \);

(ii) \( tf(x, t) > 0 \) for all \( x \) and \( t \neq 0 \);

(iii) for all \( x \), \( t^{-1}f(x, t) \) is an increasing function of \( |t| \) such that

\[
\lim_{t \to 0} t^{-1}f(x, t) = 0; \quad \lim_{t \to \infty} t^{-1}f(x, t) \geq c(x).
\]

In recent years, a great deal of effort has gone into investigating partial differential equations on fractals, see for example refs. [1—7]. One of the difficulties in studying PDEs on fractal domains is how to define differential operators, like the Laplacian, on the fractal domains. There is no concept of a generalized derivative of a function, and so we need to
clarify the notion of differential operators such as the Laplacian on fractal domains. For some special fractals, after introducing an energy form, we may define the Laplacian as a limit of difference operators. This method originates in the classical setting, that is, the Gauss average theorem, which states that a continuous function is harmonic in an open smooth domain $\Omega$ if and only if the value $u(x_0)$ of $u$ at centre $x_0$ of a ball with radius $r$ is equal to the arithmetic average of $u(x)$ on the surface $S_r = \{ x : |x-x_0| = r \}$ for all $r \leq r_0$ for some $r_0 > 0$. Thus for a discrete domain $\Omega$, if the value of $u$ at a point $x_0$ is equal to the weighted average of values of $u$ at neighboring points, we can say that $u$ is “harmonic” on $\Omega$. This idea yields the definition of harmonic function on certain classes of fractals, for example, on the Sierpiński gasket.

Another direct approach to defining the Laplacian on the Sierpiński gasket was given by Kozlov from the point of view of functional analysis, in which the pre-Sierpiński gasket is viewed as the resistor network and then the Joule heat and an energy form are well-defined on the pre-Sierpiński gasket.

Once a Laplacian is defined on a fractal, we may introduce a Hilbert space structure and then establish compactness theorems through which (1.1) can be investigated.

In this paper we work on a specific fractal, the Sierpiński gasket $K$ in $\mathbb{R}^2$, which is typical of the more general class of post-critically-finite fractals. We will establish the existence of non-trivial solutions to (1.1) on $K$. In Section 2 we review the definition of Laplacian on the Sierpiński gasket $K$ given in refs. [2, 5]. To do this we define an energy form leading to a Hilbert space $H^1_0(K)$ of functions of finite energy. Due to the construction of the Sierpiński gasket, there is a Sobolev-like inequality on $K$, see (2.5), which plays a significant role in our analysis. In Section 3 we prove the existence of multiple non-trivial solutions to (1.1) for $f(x,t)$ satisfying the conditions (i)—(iii) and $c(x)$ suitably large. To do this, we observe that the corresponding eigenvalue problem

$$\Delta u + \lambda c(x) u = 0,$$

has a sequence of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_n \to \infty, n \to \infty,$$

and the corresponding eigenfunctions $\{ \varphi_n \}$ form a basis of the Hilbert space $H^1_0(K)$, see refs. [6, 8—10]. Moreover, the Weyl-type estimate

$$\# \{ \text{eigenvalues } \leq \lambda \} \sim \lambda^{d_s/2},$$

holds for large $\lambda$, where $d_s = \frac{2 \log 3}{\log 5}$ is the spectral dimension of the Sierpiński gasket, see refs. [4, 6]. Finally, in Section 4 we show that if $f(x,t)$ satisfies a further condition (iv) below and the eigenvalue $\lambda_p < 1$ for some integer $p$, then there are $p$ pairs non-trivial solutions to (1.1). The concept of genus will be used to obtain these results. Note that we do not need any growth restriction of $f(x,t)$ in $t$, such as is required in the existence theory of non-trivial weak solutions to (1.1) on smooth domains, see refs. [11, 12]. This work is partially motivated by the classical results and methods of ref. [11], but there are considerable differences stemming from the strong Sobolev-type inequality that holds in this fractal situation.
Falconer obtained existence results in the special case where $a(x) \equiv 0$ but for more general fractal domains. It is believed that the results of this paper could generalize at least to the class of p.c.f. fractals.

2 The energy $W(u)$ and Hilbert space $H^1_0(K)$

We review the definition of the Sierpiński gasket $K$ in $\mathbb{R}^2$ and then introduce the energy $W(u)$ for $u : K \to \mathbb{R}$. From the energy $W(u)$, we construct a Hilbert space $H^1_0(K)$, leading to a striking embedding result (2.3), a consequence of the construction of the Sierpiński gasket.

We first recall the definition of the Sierpiński gasket in $\mathbb{R}^2$. Let $p_1, p_2, p_3 \in \mathbb{R}^2$ satisfy $|p_i - p_j| = 1$ for $i \neq j$. Define

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i, \quad i = 1, 2, 3,$$

and for each word $w = w_1 w_2 \cdots w_m \in \{1, 2, 3\}^m$ and define the $m$-th approximation to the Sierpiński gasket by

$$K_m = \bigcup_{w \in \{1, 2, 3\}^m} F_w(p_1, p_2, p_3),$$

for $m \geq 0$. The Sierpiński gasket $K$ is defined as the closure of $K_\star = \bigcup_{m \geq 0} K_m$ in $\mathbb{R}^2$. The set $K_0 = \{p_1, p_2, p_3\}$ is called the boundary of $K$.

For all $u : K_\star \to \mathbb{R}$, we define

$$W_m(u) = \left(\frac{5}{3}\right)^m \sum_{y \in K_m} (u(y) - u(x))^2,$$  \hspace{1cm} (2.1)

for $m \geq 0$, see ref. [11]. For all $u : K_\star \to \mathbb{R}$, the energy $W(u)$ is defined as

$$W(u) = \lim_{m \to \infty} W_m(u),$$

(possibly $W(u) = +\infty$). The following lemma justifies the above definition of $W(u)$ for all $u : K_\star \to \mathbb{R}$.

**Lemma 2.1.** For all $u : K_\star \to \mathbb{R}$, $W_m(u)$ defined by (2.1) is nondecreasing in $m \geq 0$.

**Proof.** This can be easily checked by induction and looking for the minimum of a quadratic function, see ref. [5]. We omit the detail. Q.E.D.

We also need the following

**Lemma 2.2** (Sobolev-type inequality). For all continuous functions $u : K \to \mathbb{R}$,

$$\sup_{x, y \in K_\star, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 9 \sqrt{W(u)},$$

(2.3)

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where \( \alpha = \frac{\nu g(\frac{3}{2})}{2 \log 2} \).

**Proof.** Without loss of generality, we suppose \( W(u) = \sup_n W_n(u) < \infty \) and \( |x - y| < 1 \). For such \( x, y \in K_\ast \), there exists an integer \( m = m(x, y) \) depending on \( x, y \) such that
\[
2^{-(m+1)} \leq |x - y| < 2^{-m}. \tag{2.4}
\]

We can choose a sequence of points \( \{x_n\}_{n=0}^\infty \) satisfying \( x_n \to x \) as \( n \to \infty \) and \( x_n, x_{n+1} \in K_{m+1+n} \), 
\[
|x_{n+1} - x_n| = 2^{-(m+1+n)}, \quad n = 0, 1 \ldots
\]

From the definition of \( W_n(u) \),
\[
|u(x_{n+1}) - u(x_n)| \leq \left( 3 \right)^{\frac{n+1}{2}} \frac{1}{1 - \sqrt{\frac{3}{5}}} \sqrt{W_{m+1+n}(u)}, \quad n \geq 0,
\]
and so
\[
|u(x) - u(x_0)| = \lim_{n \to \infty} |u(x_{n+1}) - u(x_0)| \\
\leq \lim_{n \to \infty} \sum_{j=0}^n |u(x_{j+1}) - u(x_j)| \\
\leq \left( 3 \right)^{\frac{n+1}{2}} \frac{1}{1 - \sqrt{\frac{3}{5}}} \sqrt{W(u)}. \tag{2.5}
\]

Similarly, we have that
\[
|u(y) - u(y_0)| \leq \left( 3 \right)^{\frac{n+1}{2}} \frac{1}{1 - \sqrt{\frac{3}{5}}} \sqrt{W(u)}. \tag{2.6}
\]
where we can choose \( x_0 = y_0 \). Combining (2.5), (2.6), it follows that
\[
|u(x) - u(y)| \leq \frac{2}{1 - \sqrt{\frac{3}{5}}} \left( 3 \right)^{\frac{n+1}{2}} \sqrt{W(u)},
\]
which combines with (2.4) to give
\[
|u(x) - u(y)| \leq \frac{2}{1 - \sqrt{\frac{3}{5}}} \frac{|x - y|^\alpha \sqrt{W(u)}}{\alpha} \leq \frac{9|x - y|^\alpha \sqrt{W(u)}}{\alpha}.
\]

Q.E.D.

We will see that (2.3) plays a central role in our analysis below. In ref. [5], a similar result to (2.3) was presented, but the proof was omitted.

Let \( C(K) \) be the space of continuous functions on \( K \) and \( C_0(K) = \{ u | u \in C(K) \text{ and } u|_{K_0} = 0 \} \). We define
\[
H^1_0(K) = \{ u | u \in C_0(K), W(u) < \infty \},
\]
which is endowed with the norm
\[
\| u \| = \sqrt{W(u)}. \tag{2.7}
\]
Setting
\[
W_m(u, v) = \left( \frac{5}{3} \right)^m \sum_{|x-y|=2^{-m}} (u(x) - u(y))(v(x) - v(y)),
\]

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it is easy to see using Cauchy’s inequality that
\[ W(u, v) = \lim_{m \to \infty} W_m(u, v) \]
events and is finite if \( u, v \in H_0^1(K) \). One can verify that \( H_0^1(K) \) with the inner product \( W(\cdot, \cdot) \) is a real Hilbert space.

By (2.3), for all \( u \in H_0^1(K) \)
\[ |u(x) - u(y)| \leq 9|x - y|\|u\|, \]
and taking \( y = p_1 \) we readily get
\[ |u(x)| \leq 9\|u\|. \tag{2.8} \]

**Remark 2.1.** From Lemma 2.2, the embedding
\[ H_0^1(K) \hookrightarrow C_0(K), \tag{2.9} \]
is compact.

We now define Laplacian on the Sierpiński gasket \( K \). Let \( H^{-1}(K) \) be the closure of \( L^2(K) \) with respect to the pre-norm
\[ \|v\|_{-1} = \sup_{g \in H_0^1(K)} |\langle v, g \rangle|, \]
where
\[ \langle v, g \rangle = \int_K vgd\mu, \]
for \( v \in L^2(K) \) and \( g \in H_0^1(K) \). Then \( H^{-1}(K) \) is a Hilbert space. Let \( W(u, v) \) be the inner product of \( u, v \in H_0^1(K) \). Then the relation
\[-W(u, v) = \langle \Delta u, v \rangle, \quad \text{for all } v \in H_0^1(K), \tag{2.10} \]
uniquely defines a function \( \Delta u \in H^{-1}(K) \) for all \( u \in H_0^1(K) \); we term \( \Delta \) the (weak) Laplacian on \( K \), see ref. [5].

We say \( u \in H_0^1(K) \) is a weak solution to (1.1) if
\[ W(u, v) - \int_K c(x)uvd\mu + \int_K f(x, u)vvd\mu = 0 \quad \text{for all } v \in H_0^1(K). \tag{2.11} \]

Whilst we shall mainly work with the weak Laplacian, there is also a directly defined version by Kigami. We say that \( \Delta_s u \) is the standard Laplacian of \( u \) if \( \Delta_s : K \to \mathbb{R} \) is continuous and
\[ \lim_{m \to \infty} \sup_{x \in K \setminus K_m} \left| \frac{3}{2} 5^m(H_m u)(x) - \Delta_s u(x) \right| = 0, \]
where
\[ (H_m u)(x) = \sum_{y \in \hat{K}_m} (u(y) - u(x)), \]
for \( x \in K_m \), see ref. [2]. We say that \( u \in C_0(K) \) is a strong solution of (1.1) if \( \Delta_s u(x) \) exists and is continuous for all \( x \in K \setminus K_0 \), and
\[ \Delta_s u(x) = f(x, u(x)) - c(x)u(x), \quad \text{for all } x \in K \setminus K_0. \]
The main results of this paper are

**Theorem 2.1.** Let \( c(x) \geq 0 \) with the property that \( \int_K c(x) \, d\mu > 0 \) and \( f(x, t) \) satisfy (i)—(iii). Then (1.1) has non-trivial weak solutions \( \pm u_0(x) \) if and only if \( \lambda_1 < 1 \). If \( \lambda_1 \geq 1 \), (1.1) has only the zero solution on \( K \).

**Theorem 2.2.** Let \( c(x) \geq 0 \) with the property that \( \int_K c(x) \, d\mu > 0 \) and \( f(x, t) \) satisfy (i)—(iii) and further

\[
\frac{\partial f}{\partial t} (x, t) > t^{-1} f(x, t), \quad \text{for all } x \text{ and } t \neq 0,
\]

and \( f(x, t) := tf(x, t) \) is locally Lipschitz continuous in \( t \) uniformly for \( x \). Then (1.1) has \( p \) pairs of non-trivial weak solutions \( \pm u(x) \) if \( \lambda_p < 1 \leq \lambda_{p+1} \) for some integer \( p \).

We mention that if \( f \) and \( c \) are continuous on \( K \), all weak solutions of (1.1) obtained in Theorems 2.1 and 2.2 are essentially (within measure 0) strong solutions, see the discussion in ref. [15].

### 3 Proof of Theorem 2.1

Let \( \Gamma: H^1_0(K) \to \mathbb{R} \) be the functional defined on the Hilbert space \( H^1_0(K) \)

\[
\Gamma(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \langle Cu, u \rangle + h(u), \tag{3.1}
\]

where

\[
\langle Cu, v \rangle = \int_K c(x) u(x)v(x) \, d\mu, \\
h(u) = \int_0^1 ds \int_K f(x, su(x))u(x) \, d\mu = \int_0^1 \langle Fsu, u \rangle \, d\mu, \tag{3.2}
\]

\[
\langle Fu, v \rangle = \int_K f(x, u(x))v(x) \, d\mu.
\]

From conditions (i)—(iii), we readily deduce that

(F.i) \( F(-u) = F(u) \) for all \( u \in H^1_0(K) \);

(F.ii) \( \langle Fu, u \rangle > 0 \) for all \( u \neq 0 \);

(F.iii) for fixed \( u \neq 0 \), \( s^{-1}\langle Fsu, u \rangle \) is an increasing function of \( s > 0 \) such that

\[
\lim_{s \to 0} s^{-1}\langle Fsu, u \rangle = 0; \quad \lim_{s \to \infty} s^{-1}\langle Fsu, u \rangle \geq \langle Cu, u \rangle.
\]

Also

(h.i) \( h(-u) = h(u) \) for all \( u \in H^1_0(K) \);

(h.ii) \( h(u) \geq 0 \) and \( h(u) = 0 \) if and only if \( u = 0 \);

(h.iii) For fixed \( u \neq 0 \), \( s^{-2}h(su) \) is an increasing function of \( s > 0 \) such that

\[
\lim_{s \to 0} s^{-2}h(su) = 0; \quad \lim_{s \to \infty} s^{-2}h(su) \geq \frac{1}{2}\langle Cu, u \rangle.
\]
To prove Theorem 2.1, we express (1.1) in the following variational form.

**Proposition 3.1.** (2.11) is the Euler equation for the functional \( \Gamma(u) \), that is, a function \( u \in H_0^1(K) \) satisfies (2.11) if and only if the Fréchet derivative \( \Gamma'(u) \) of the functional \( \Gamma(u) \) is equal to zero.

**Proof.** We first show that for any \( u \in H_0^1(K) \), the Fréchet derivative of the functional \( h(u) \) is equal to \( Fu \). For simplicity, we write \( f'(x,t) = \frac{\partial f}{\partial t}(x,t) \). For \( u, v \in H_0^1(K) \), by (3.2) and (2.8),

\[
|h(u + v) - h(u) - \langle Fu, v \rangle| = |\int_0^1 ds \int_K [f(x, s(u + v))(u + v) - f(x, su)u]d\mu
- \int_K v d\mu \int_0^1 \frac{d}{ds}(f(x, su)s)ds|
= |\int_0^1 ds \int_K [(f'(x, su + \theta_1 sv) - f'(x, su))su]d\mu
- f'(x, su)du + f'(x, su + \theta_2 sv)sv^2)d\mu|
\leq b\|v\| \int_0^1 ds \int_K [(f'(x, su + \theta_1 sv) - f'(x, su))\|u\|
+ \|f'(x, su + \theta_2 sv)\|v\|)d\mu,
\]

where \( 0 \leq \theta_1, \theta_2 \leq 1 \) and \( b \) is a constant independent of \( u \) and \( v \).

Therefore,

\[
\lim_{\|v\| \to 0} \frac{|h(u + v) - h(u) - \langle Fu, v \rangle|}{\|v\|} = 0,
\]

since \( f'(x,t) \) is continuous in \( t \) uniformly for all \( x \in K \). Thus \( h(u) \) has Fréchet derivative \( Fu \) at \( u \in H_0^1(K) \). Proposition 3.1 follows by noting that

\[
\langle \Gamma'(u), v \rangle = W(u, v) - \langle Cu, v \rangle + \langle Fu, v \rangle \quad \text{for all} \quad v \in H_0^1(K).
\]

Q.E.D.

We note that if the functional \( \Gamma(u) \) reaches its minimum or maximum at \( u_0 \), then \( \Gamma'(u_0) = 0 \).

**Proof of Theorem 2.1.** We first claim that \( \Gamma(u) \) given by (3.1) is bounded from below. Otherwise, there is a sequence \( \{u_n\} \subset H_0^1(K) \) such that \( \Gamma(u_n) \to -\infty \). We write \( u_n \) as

\[
u_n = \sum_{k=1}^{\infty} \alpha_n^k \varphi_k, \quad \alpha_n^k \in \mathbb{R},
\]

where \( \{\varphi_k\}_{k=1}^{\infty} \) is the basis in \( H_0^1(K) \) of eigenfunctions \( \varphi_k \) corresponding eigenvalues \( \lambda_k \), see (1.2). Thus

\[
\|u_n\|^2 \geq \lambda_1 \langle Cu_n, u_n \rangle
= \lambda_1 (\|u_n\|^2 + 2h(u_n) - 2\Gamma(u_n))
\geq -2\lambda_1 \Gamma(u_n) \to \infty \quad \text{as} \quad n \to \infty.
\]
Let $s_n = \frac{1}{\|u_n\|} \to 0$ as $n \to \infty$, we can suppose that $s_n \leq b^{-1}$ for $b > 0$. Since $\{s_n u_n\}$ is bounded in $H^1_0(K)$, there exist a subsequence (which, without loss of generality, we denote by $\{s_n u_n\}$) and $w(x) \in C_0(K)$ such that $s_n u_n(x) \to w(x)$ as $n \to \infty$ in $C_0(K)$ in virtue of (2.9) and (2.8). We then have by (3.1) and (h.iii), since $b s_n \leq 1$,

$$0 \geq 2 s_n^2 \Gamma(u_n)
= 1 - s_n^2 \langle Cu_n, u_n \rangle + 2 s_n^2 h(u_n)
\geq 1 - s_n^2 \langle Cu_n, u_n \rangle + 2 b^{-2} h(s_n b u_n).$$

(3.3)

Letting $n \to \infty$ in (3.3) and using the Lebesgue dominated convergence theorem, it follows that

$$1 - \langle Cu, w \rangle + 2 b^{-2} h(b w) \leq 0.$$ 

(3.4)

But by (h.iii), $2 b^{-2} h(b w) \geq \langle Cu, w \rangle - \frac{1}{2}$ for $b$ large enough, and thus (3.4) is impossible. Hence $\Gamma(u)$ is bounded from below.

Next we show that there exists $u_0 \in H^1_0(K)$ such that $\Gamma(u)$ reaches its minimum at $u_0$. Indeed, let $\{u_n\}$ be a minimising sequence for $\Gamma$ and $\sigma = \inf_{u \in H^1_0(K)} \Gamma(u)$. Then $\sigma \leq 0$. As in the above argument, we have that $\{u_n\}$ is bounded in $H^1_0(K)$ and thus by Remark 2.1 there exists $u_0 \in C_0(K)$ such that without loss of generality, $u_n(x) \to u_0(x)$ as $n \to \infty$ in $C_0(K)$. Note that

$$\|u_0\|^2 = \lim_{n \to \infty} \lim_{m \to \infty} W_m(u_n)
\leq \lim_{n \to \infty} W(u_n) = \lim_{n \to \infty} \|u_n\|^2,$$ 

(3.5)

so $u_0 \in H^1_0(K)$.

By (3.5) and using again the Lebesgue dominated convergence theorem, it follows that

$$\sigma \leq \Gamma(u_0) = \frac{1}{2} \|u_0\|^2 - \frac{1}{2} \langle Cu_0, u_0 \rangle + h(u_0)
\leq \lim_{n \to \infty} \left[ \frac{1}{2} \|u_n\|^2 - \frac{1}{2} \langle Cu_n, u_n \rangle + h(u_n) \right]
= \lim_{n \to \infty} \Gamma(u_n) = \sigma,$$

and hence $\Gamma(u_0) = \sigma$.

Finally, we claim that if $\lambda_1 < 1$ then $\sigma < 0$; and if $\lambda_1 \geq 1$ then $\sigma = 0$ and $u_0 = 0$.

To see this, suppose $\lambda_1 < 1$ and let $\varphi_1$ be the normalized eigenfunction so that $1 = \|\varphi_1\|^2 = \lambda_1 \langle C \varphi_1, \varphi_1 \rangle$. Then

$$2 \Gamma(s \varphi_1) = 2 h(s \varphi_1) - s^2 \langle C \varphi_1, \varphi_1 \rangle - \|\varphi_1\|^2
= 2 h(s \varphi_1) - s^2 (\lambda_1^{-1} - 1)
= s^2 (2 s^{-2} h(s \varphi_1) - (\lambda_1^{-1} - 1)).$$

By (h.iii), we can find $s > 0$ such that $\Gamma(s \varphi_1) < 0$ and hence $\sigma < 0$.

On the other hand, if $\lambda_1 \geq 1$, then for any $u \in H^1_0(K)$

$$\|u\|^2 \geq \lambda_1 \langle Cu, u \rangle \geq \langle Cu, u \rangle,$$
and so, by (h.ii), $\Gamma(u) \geq 0$ and thus $\sigma = 0$. Therefore $u_0 = 0$.

Note that the minimizing sequence may be chosen from those functions in $H^1_0(K)$ which are non-negative on $K$ since $\Gamma(-u) = \Gamma(u)$. By Proposition 3.1, the $u_0$ obtained above is a weak solution to (1.1).

**Remark 3.1.** Taking $c(x) \equiv \lambda$ in (1.1), from (2.8) it follows that

$$
\lambda_1 = \inf_{\varphi \in H^1_0(K)} \frac{\|u\|^2}{\lambda(u, u)} \geq \frac{1}{81\lambda}.
$$

Thus if $0 < \lambda \leq \frac{1}{81}$, then $\lambda_1 \geq 1$, and thus

$$
\triangle u + \lambda u = f(x, u)
$$

has no non-trivial solutions in the Hilbert space $H^1_0(K)$ on the Sierpiński gasket provided that $f(x, t)$ satisfies (i)–(iii), in particular, provided that $f(x, u) = u|u|^{p-1}$ for all $p > 1$.

### 4 Proof of Theorem 2.2

**Preliminary 4.1.** Let $E$ be a real Banach space and $\sum(E)$ the class of closed subsets $E_i \subset E - \{0\}$ symmetric through the origin, that is, $x \in E_i$ implies $-x \in E_i$. We use the notion of genus of subsets of $\sum(E)$.

**Definition 4.1.** $E_i \in \sum(E)$ has genus $n$ (denoted by $\rho[E_i] = n$) if $n$ is the maximum integer for which every odd continuous map of $E_i$ into $\mathbb{R}^{n-1}$ has a zero on $E_i$. If $E_i$ is empty its genus is defined to be zero.

The following properties are standard:

1. if $E_i$ is homoeomorphic to an $n-1$ sphere $S^{n-1} = \{x \in \mathbb{R}^{n-1}: \|x\| = 1\}$, then $\rho[E_i] = n$ and $E_i$ contains subsets of every genus less than $n$.
2. if $E_i \subset E_2$, then $\rho[E_i] \leq \rho[E_2]$.
3. if $\phi$ is an odd continuous map of $E_i$, then $\rho[E_i] \leq \rho[\phi(E_i)]$.
4. every $E_i$ has a neighborhood $N$ in $E$ such that $\rho[E_i] = \rho[N]$.
5. $\rho[E_1 \cup E_2] \leq \rho[E_1] + \rho[E_2]$.

For more details we refer the reader to refs. [8, 12] and references therein. We note that an $n-1$ sphere $S^{n-1}$ has genus $n$ but its Euclidean dimension is $n-1$. Because of property (3), the notion of genus is very useful in connection with certain odd continuous maps such as the projection and radial maps.

**Remark 4.1.** Let $E_1 \subset E$ be closed with $E_1 \cap (-E_1) = \emptyset$. Let $E_2 = E_1 \cup (-E_1)$. Then $\rho[E_2] = 1$ since the function $\varphi(x) = 1$ for $x \in E_1$ and $\varphi(x) = -1$ for $x \in -E_1$ is odd and continuous, and has no zero point in $E_2$. Therefore, if $E_2 \subset \sum(E)$ and $\rho[E_2] > 1$ then $E_2$ contains infinitely many distinct points, for otherwise we could write $E_2 = E_1 \cup (-E_1)$ with $E_1$ closed and $E_1 \cap (-E_1) = \emptyset$, and $\rho[E_2] = 1$.
Next we observe that (2.11) is equivalent to the following operator equation in \( H^1_0(K) \)

\[
Au + Fu = Cu,
\]

where \( C \) and \( F \) are given in (3.2) and \( A: H^1_0(K) \to (H^1_0(K))^* \) (dual of \( H^1_0(K) \)), i.e. given by

\[
\langle Au, v \rangle = W(u, v), \quad \text{for all } v \in H^1_0(K),
\]

where \( \langle Au, v \rangle \) means the value of the functional \( Au \) at point \( v \). It is easily seen from (4.1) that \( A \) is linear, bounded and self-adjoint on \( H^1_0(K) \). The operator \( C \) is compact, positive and self-adjoint while \( F \) is compact by (2.9), (2.8). Moreover, \( F \) has further useful properties given in the following proposition.

**Proposition 4.1.** Assume that \( f \) satisfies (i)—(iv). Then the operator \( F \) is uniformly Lipschitz continuous on bounded subsets of \( H^1_0(K) \). The functional \( T: H^1_0(K) \to \mathbb{R} \) defined by \( T(u) = \frac{1}{2} \langle Fu, u \rangle \) has a Fréchet derivative \( B \) which is also uniformly Lipschitz continuous on bounded subsets of \( H^1_0(K) \). Furthermore, for all \( u \neq 0 \),

\[
\langle Bu, u \rangle > \langle Fu, u \rangle.
\]

**Proof.** Let \( f'(x, t) = \frac{\partial f}{\partial t}(x, t) \) and \( B \) be the operator on \( H^1_0(K) \) given by

\[
\langle Bu, v \rangle = \frac{1}{2} \int_K (f'(x, u)u + f(x, u))v \, d\mu, \quad \text{for all } v \in H^1_0(K).
\]

As in the proof of Proposition 3.1, it is easy to check that the functional \( T(u) \) has the Fréchet derivative \( Bu \) at every \( u \in H^1_0(K) \). Moreover, since \( f(x, t) = tf(x, t) \) is uniformly Lipschitz in \( t \) for all \( x \), by (4.3), if \( E_1 \) is a bounded subset of \( H^1_0(K) \) and \( u_1, u_2 \in E_1 \) and \( v \in H^1_0(K) \),

\[
|\langle Bu_2 - Bu_1, v \rangle| = \frac{1}{2} \left| \int_K ((f(x, u_2) - f(x, u_1)) + (f(x, u_2) - f(x, u_1))) v \, d\mu \right| \leq b \|v\| \|u_2 - u_1\|,
\]

where the constant \( b \) depends only on \( E_1 \). Hence \( B \) satisfies a Lipschitz condition. By (4.3) and (iv),

\[
\langle Bu, u \rangle - \langle Fu, u \rangle = \frac{1}{2} \int_K (f'(x, u)u - f(x, u)) \, d\mu = \frac{1}{2} \int_K \left( f'(x, u) - \frac{f(x, u)}{u} \right) u^2 \, d\mu > 0, \quad u \neq 0,
\]

and so (4.2) holds. That \( Fu \) is uniformly Lipschitz continuous on bounded subsets of \( H^1_0(K) \) can be verified in a similar way. Q.E.D.

**Definition 4.2.** We define the set \( S \) by

\[
S = \{ u | u \in H^1_0(K), \langle Au, u \rangle + \langle Fu, u \rangle = \langle Cu, u \rangle, u \neq 0 \}.
\]

Any non-trivial solution to (1.1) must belong to \( S \). We have
Proposition 4.2.  (1) $S$ is a bounded subset of $H^1_0(K)$.

(2) If $\lambda_p < 1 \leq \lambda_{p+1}$, then $S$ contains compact symmetric subsets of every genus $\leq p$ and no such subsets of genus $> p$.

Proof.  (1) If $S$ is unbounded, then there is a sequence $\{u_n\} \subset S$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Let $s_n = \frac{1}{\|u_n\|}$, and so $s_n \to 0$ as $n \to \infty$. By (2.9) we may assume on taking a subsequence, $s_n u_n \to w$ in $C_0(K)$. As in (3.5), $\|w\| \leq \lim_{n \to \infty} \|s_n u_n\| = 1$, so $w \in H^1_0(K)$. As in the proof of Theorem 2.1 this leads to a contradiction. Therefore $S$ is a bounded subset of $H^1_0(K)$.

Proof of (2) is a modification of that of Lemma 3.2 in ref. [11]. We sketch the proof.

Let $E_m = \text{span}\{\varphi_1, \cdots, \varphi_m\}$ be the space composed of the eigenfunctions $\varphi_m$ of (1.2) corresponding to the eigenvalues $\lambda_m$, $1 \leq m \leq p$. Then for $u \in E_m(u \neq 0)$, we have $\langle Au, u \rangle < \langle Cu, u \rangle$ since $\lambda_p < 1$, and by (F.iii) there is a unique $s = s(u) > 0$ which is continuous in $u$ such that

$$s^{-1} \langle Fu, u \rangle = \langle Cu, u \rangle - \langle Au, u \rangle. \tag{4.5}$$

Thus $su \in S$. We claim that $\rho[S \cap E_m] = m, 1 \leq m \leq p$.

It is easily seen that $S \cap E_m$ is closed, symmetric through the origin and contains no zero. Define $\phi_1 : S \cap E_m \to S^{m-1} = \{u \in E_m : \|u\| = 1\}$ by $\phi_1(u) = \frac{u}{\|u\|}$. Since $\phi_1$ is odd and continuous and $\rho[S^{m-1}] = m$, by property (3) of genus, we have $\rho[S \cap E_m] \leq \rho[S^{m-1}] = m$. On the other hand, let $\phi_2 : S^{m-1} \to S \cap E_m$ be defined by $\phi_2(u) = s(u)u$. Here $s(u)$ is given by (4.5) and $s(-u) = s(u)$, for all $0 \neq u \in E_m$. Thus, $\phi_2(-u) = -\phi_2(u)$, and so $\phi_2$ is odd and continuous. Again by property (3) of genus, $\rho[S \cap E_m] \geq \rho[S^{m-1}] = m$. Therefore, $\rho[S \cap E_m] = m, 1 \leq m \leq p$, and so $S$ contains compact subsets of every genus $m \leq p$.

Assume that there is a set $E \subset S$ with $\rho[E] \geq p + 1$. We decompose $u \in E$ as $u = u_1 + u_2$, $u_1 \in E_p, u_2 \in E_p^\perp$.

Write $u_1 = \sum_{k=1}^p \alpha_k \varphi_k \alpha_k \in \mathbb{R}$ and let $\phi(u) : E \to \mathbb{R}^p$ be the map given by $\phi(u) = (\alpha_1, \cdots, \alpha_p)$. Then $\phi$ is odd and continuous. Since $\rho[E] \geq p + 1$, by Definition 4.1 there is a point $w \in E$ such that $\phi(w) = 0$, so $w \in E_p^\perp$. But for such $w$ we have

$$\langle Aw, w \rangle \geq \Lambda_{p+1} \langle Cw, w \rangle \geq \langle Cw, w \rangle,$$

and so

$$\langle Fw, w \rangle = \langle Cw, w \rangle - \langle Aw, w \rangle \leq 0.$$

This is only true for $w = 0$, leading to a contradiction, since $w \in S$. Q.E.D.

Proof of Theorem 2.2.  Let $\Gamma_i$ be the class of compact symmetric subsets $E$ of $S$ with $\rho[E] \geq i$ and define the numbers $\Lambda_i$ by

$$\Lambda_i = \sup_{E \in \Gamma_i} \min_{u \in E} g(u), \tag{4.6}$$

where $g(u) = \frac{1}{2} \langle Fu, u \rangle - h(u)$. Then $\Lambda_i$ is well defined for $1 \leq i \leq p$ since $\Gamma_i$ is non-empty, but $\Lambda_i$ is not well defined if $i \geq p$ since $\Gamma_i$ is empty if $i \geq p$, by Proposition
4.2. Moreover we have
\(\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_p > 0.\)  
\hfill (4.7)

Indeed, for \(u \in S \cap E_p \subset \Gamma_p\), conditions (i) and (iii) imply \(g(u) > 0\). Since \(g(u)\)

is continuous on \(H^1_0(K)\) and \(S \cap E_p\) is closed, we must have \(\min_{u \in S \cap E_p} g(u) > 0\) so \(\Lambda_p > 0\). By (4.6), \(\Lambda_i\) is clearly non-increasing in \(i\).

We now consider the initial value problem
\[
\frac{dw}{dt} = Bw - Fw - \xi(w)(Aw - Cw + Bw),
\]
\[
w(0) = u,
\]  
for any \(u \in S\), where \(\xi(w) = \|Aw - Cw + Bw\|^{-2}(Aw - Cw + Bw, Bw - Fw)\), and \((\cdot, \cdot)\) denotes the inner product in the Hilbert space \(H^1_0(K)\). Note that \(Aw, Cw, Bw\) and \(Fw\) can be viewed as the elements of the Hilbert space \(H^1_0(K)\) for any \(w \in H^1_0(K)\), thanks to the Riesz Representation Theorem. Thus (4.8) is an ordinary differential equation on the abstract space \(H^1_0(K)\) with the initial condition \(w(0) = u\). For convenience, we rewrite (4.8) as
\[
\frac{dw}{dt} = Gw - (Gw, Iw)Iw,
\]
\[
w(0) = u,
\]  
where \(Gw = Bw - Fw\) and \(Iw = \|Aw - Cw + Bw\|^{-1}(Aw - Cw + Bw)\). For any \(w \in S\), \(Iw\) is well defined since, by (4.2),
\[
(Aw - Cw + Bw, w) = (Bw, w) - (Fw, w) > 0.
\]

Let \(Lw = Aw - Cw + Bw\). By (2.8), (2.9) and (4.2), we have
\[
\inf_{w \in D} \|Lw\| \geq b > 0,
\]
where \(D\) is a given closed subset of \(S\) and where \(b\) depends only on \(D\). It is easily seen by

Proposition 4.1 that \((Gw, Iw)Iw\) is uniformly Lipschitz on closed subsets of \(S\). Hence the right side of (4.9) is uniformly Lipschitz on closed subsets of \(S\). Obviously, the right side of (4.9) is uniformly bounded on \(S\) since \(S\) is bounded by Proposition 4.2. Thus we have shown

**Lemma 4.1.** Let \(A, C, B\) and \(F\) be the operators on \(H^1_0(K)\) and \(\xi(w) \in \mathbb{R}\) defined above. Then for all \(u \in H^1_0(K)\), the initial value problem (4.9) possesses a unique solution \(w(t, u)\) for \(t \in (-\infty, \infty)\).

Note that \(w(t, u) \in S\) for \(t \geq 0\) since, by (4.9), \((\frac{dw}{dt}, Iw) = 0\) and so
\[
\frac{d}{dt} (Aw - Cw + Fw, w) = 0,
\]
which reduces to
\[
(Aw - Cw + Fw, w) = (Au - Cu + Fu, u) = 0.
\]

Moreover, by uniqueness of solutions to (4.9),
\[
w(t, -u) = -w(t, u), \quad t \in (-\infty, \infty).
\]  
\hfill (4.10)
Next, if \( w \) is a solution to (4.9), we have that
\[
\frac{d}{dt}g(w) = \left< g'(w), \frac{dw}{dt} \right> = \left< Gw, \frac{dw}{dt} \right> = (Gw, Gw) - (Gw, Iw)^2
\]
and therefore
\[
(Gw - (Gw, Iw)Iw, Gw - (Gw, Iw)Iw) = \left\| \frac{dw}{dt} \right\|^2 \geq 0.
\]  
(4.11)
Thus \( g(w) \) is non-decreasing along trajectories of (4.9).

We claim that the set of stationary points of (4.9) coincides with the set of non-trivial solutions to (4.1). To show this, suppose \( w \in S \) and
\[
Bw - Fw = \xi(w)(Aw - Cw + Bw).
\]
It follows from (4.2) that
\[
0 < (Bw, w) - (Fw, w) = \xi(w)(Aw - Cw + Fw, w) = \xi(w)((Bw, w) - (Fw, w)),
\]
and so \( \xi(w) = 1 \) and \( Aw + Fw = Cw \). Conversely, if \( Aw + Fw = Cw \), we have \( \xi(w) = 1 \) and \( \frac{dw}{dt} = 0 \).

For \( 1 \leq i \leq p \), we define \( Q_i \) by
\[
Q_i = \{ u : Au + Fu = Cu, g(u) = \Lambda_i \},
\]
then \( Q_i \) is a (possibly empty) compact symmetric subset of \( S \) and has the following property.

**Lemma 4.2.** Let \( N \) be a neighborhood such that \( Q_i \subset N \subset S \). Then there exists \( \eta > 0 \) such that \( Q_i \subset N_{\eta} \subset S \), where
\[
N_{\eta} = \{ u : u \in S, |g(u) - \Lambda_i| < \eta \quad \text{and} \quad \left\| \frac{dw}{dt}(t, u) \right\| < \eta \quad \text{for some} \quad t \in [0, 1] \}.
\]

**Proof.** Suppose that there is a sequence \( \{ u_n \} \subset S - N \) and a sequence \( \{ t_n \} \subset [0, 1] \) such that \( g(u_n) \to \Lambda_i \) and \( \frac{dw}{dt}(t_n, u_n) \to 0 \) as \( n \to \infty \). Let \( w_n = w(t_n, u_n) \). Since \( w_n \in S \) and therefore \( \{ u_n \} \) is bounded in \( H^1_0(K) \) by Proposition 4.2, we may assume \( t_n \to t_0 \) and \( w_n \to \bar{w}, Iw_n \to v \) weakly in \( H^1_0(K) \) and \( \| Aw_n - Cw_n + Bw_n \| \to r \).

Since \( g \) is compact, by (4.11),
\[
g(\bar{w}) = \lim_{n \to \infty} g(u_n) \geq \lim_{n \to \infty} g(u_n) = \Lambda_i > 0,
\]
and so \( \bar{w} \neq 0 \). Using (4.9) and \( \frac{dw}{dt}(t_n, u_n) \to 0 \) together with the compactness of \( G \), we have
\[
G\bar{w} = (G\bar{w}, v),
\]
and thus \( \| v \| = 1 \).

Therefore, \( Iw_n \) converges strongly to \( v \) in \( H^1_0(K) \). Hence by definition of \( I \), \( Aw_n - Cw_n + Bw_n \to rv \), and so \( w_n \) converges strongly to \( \bar{w} \) in the Hilbert space \( H^1_0(K) \), and \( v = I\bar{w} \). It follows from (4.13) that \( G\bar{w} = (G\bar{w}, I\bar{w})I\bar{w} \), that is, \( \bar{w} \) is a stationary point of (4.9).

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By uniqueness theorem for solutions of (4.9), we get that
\[ \lim_{n \to \infty} u_n = \lim_{n \to \infty} w(w_n, -t_n) = w(\bar{\omega}, -t_0) = \bar{\omega}. \]
Therefore, \( g(\bar{\omega}) = \Lambda_i \) and so \( \bar{\omega} \in Q_i \subset N \). But this is impossible since \( u_n \in S - N \) for all \( n \) and \( S - N \) is closed.

In order to prove Theorem 2.2, it is sufficient to prove

**Proposition 4.3.** For each \( i \) with \( 1 \leq i \leq p \), and let \( Q_i \) be given by (4.12). Then \( \rho[Q_i] \geq \) multiplicity of \( \Lambda_i \).

**Proof.** Let \( N \) be a neighborhood of \( Q_i \) such that \( \rho[N] = \rho[Q_i] \). By Lemma 4.2, we can choose \( N \) to be of the form \( N_\eta \) for some \( 0 < \eta < 1 \). From (4.6), there is a set \( E \in \Gamma_i \) such that
\[ \min_{u \in E} g(u) \geq \Lambda_i - \frac{\eta^2}{2}. \]
We define the continuous map \( \phi : H^1_0(K) \to H^1_0(K) \) by
\[ \phi(u) = w(1, u). \]
By (4.10), \( \phi(u) \) is odd. Let \( \bar{E} = \phi(E - N) \) be the image of the set \( E - N \) under \( \phi \). Then
\[ \min_{w \in \bar{E}} g(w) \geq \Lambda_i + \frac{\eta^2}{2}. \]
To see this, we may assume that \( u \in E - N \) and \( g(u) < \Lambda_i + \eta \) (if \( g(u) \geq \Lambda_i + \eta \), then \( g(\phi(u)) \geq g(u) \geq \Lambda_i + \eta \geq \Lambda_i + \frac{\eta^2}{2} \), and (4.14) follows immediately). Then
\[ \| \frac{d\phi}{dt}(t, u) \| \geq \eta \text{ for all } t \in [0, 1], \text{ and so by } (4.11) \]
\[ g(\phi(u)) = g(u) + \int_0^1 \| \frac{d\phi}{dt}(t, u) \|^2 dt \]
\[ \geq \Lambda_i - \frac{\eta^2}{2} + \eta^2 = \Lambda_i + \frac{\eta^2}{2}, \]
giving (4.14).

Suppose now that
\[ \Lambda_j > \Lambda_{j+1} = \cdots = \Lambda_i > \Lambda_{i+1}. \]
Since \( \min_{w \in \bar{E}} g(w) > \Lambda_i = \Lambda_{j+1} \), we must have \( \rho[\bar{E}] \leq j \) (otherwise, by (4.6) \( \Lambda_{j+1} \geq \min_{w \in \bar{E}} g(w) \geq \Lambda_{j+1} \)). Since \( E \subset (E - N) \cup \bar{N} \), it follows from properties (5), (3) of genus that
\[ i \leq \rho[E] \leq \rho[E - N] + \rho[\bar{N}] \]
\[ \leq \rho[E] + \rho[\bar{N}] \]
\[ = \rho[\bar{E}] + \rho[Q_i] \leq j + \rho[Q_i]. \]
Therefore \( \rho[Q_i] \geq i - j = \text{multiplicity of } Q_i \).

**Remark 4.2.** Note that each element of \( Q_i, 1 \leq i \leq p \), is a non-trivial solution of (1.1). For each \( 1 \leq i \leq p \), by Proposition 4.3 \( \rho[Q_i] \geq 1 \). That means \( Q_i \) contains at least two non-zero elements \( u \) and \(-u\) for each \( 1 \leq i \leq p \), both of which are the
weak solutions of (1.1). Thus if (4.7) holds strictly, that is, $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_p > 0$, then by (4.12), $Q_i \cap Q_j = \phi$ if $1 \leq i \neq j \leq p$, and so (1.1) has at least $p$ pair of non-trivial solutions. On the other hand, if there exists some $1 \leq i_0 \leq p - 1$ such that $\Lambda_{i_0} = \Lambda_{i_0+1}$, then by Proposition 4.3, $\rho[Q_{i_0}] = \rho[Q_{i_0+1}] \geq 2 > 1$ and Remark 4.1 implies $Q_{i_0}$ contains infinitely many distinct elements, and so (1.1) has infinitely many distinct non-trivial solutions.

Theorem 2.2 follows immediately from Remark 4.2.

Acknowledgements  The author is grateful to Prof. K. J. Falconer for suggesting this problem and reading through this paper, and for his valuable suggestions and discussions. This work was supported partially by the National Natural Science Foundation of China (Grant No. 10371062).

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