

## $P_{4k-1}$ -factorization of complete bipartite graphs

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**Abstract** Let  $K_{m,n}$  be a complete bipartite graph with two partite sets having  $m$  and  $n$  vertices, respectively. A  $P_v$ -factorization of  $K_{m,n}$  is a set of edge-disjoint  $P_v$ -factors of  $K_{m,n}$  which partition the set of edges of  $K_{m,n}$ . When  $v$  is an even number, Wang and Ushio gave a necessary and sufficient condition for the existence of  $P_v$ -factorization of  $K_{m,n}$ . When  $v$  is an odd number, Ushio in 1993 proposed a conjecture. However, up to now we only know that Ushio Conjecture is true for  $v = 3$ . In this paper we will show that Ushio Conjecture is true when  $v = 4k - 1$ . That is, we shall prove that a necessary and sufficient condition for the existence of a  $P_{4k-1}$ -factorization of  $K_{m,n}$  is (1)  $(2k - 1)m \leq 2kn$ , (2)  $(2k - 1)n \leq 2km$ , (3)  $m + n \equiv 0 \pmod{4k - 1}$ , (4)  $(4k - 1)mn/[2(2k - 1)(m + n)]$  is an integer.

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Let  $K_{m,n}$  be a complete bipartite graph with two partite sets having  $m$  and  $n$  vertices, respectively. A subgraph  $F$  of  $K_{m,n}$  is called a spanning subgraph of  $K_{m,n}$  if  $F$  contains all the vertices of  $K_{m,n}$ . A  $G$ -factor of  $K_{m,n}$  is a spanning subgraph  $F$  of  $K_{m,n}$  such that every component of  $F$  is a  $G$ . A  $G$ -factorization of  $K_{m,n}$  is a set of edge-disjoint  $G$ -factors of  $K_{m,n}$  which partition the set of edges of  $K_{m,n}$ . The graph  $K_{m,n}$  is called  $G$ -factorizable if it has a  $G$ -factorization. In ref. [1] the  $G$ -factorization of  $K_{m,n}$  is defined as a resolvable  $(m, n, k, 1)$   $G$ -design. For graph theoretical terms see ref. [2].

$P_v$ -factorizations and  $K_{p,q}$ -factorizations of  $K_{m,n}$  have been studied by many researchers and found to have a number of applications. Especially, Yamamoto and Ushio et al.<sup>[3,4]</sup> have given some applications in  $HUBFS_2$  and  $HUBMFS_2$  schemes of database systems. There are also some known results on the existence of the  $P_v$ -factorizations and  $K_{p,q}$ -factorizations of  $K_{m,n}$ . The existence of  $P_3$ -factorization of  $K_{m,n}$  has been completely solved by Ushio<sup>[5]</sup>. Notice that a  $P_3$  is also a  $K_{1,2}$ . Since then, Ushio, Martin and Wang et al. have done some further research in  $P_v$ -factorizations and  $K_{p,q}$ -factorizations of  $K_{m,n}$ . Martin<sup>[6]</sup> investigated the  $K_{p,q}$ -factorization of  $K_{m,n}$  and gave a necessary condition for such a factorization to exist and conjectured that the condition is also sufficient. Martin Conjecture, from then on, has drawn focus from many researchers. In refs.

[7—9], we proved that Martin Conjecture is true when  $p : q = k : (k + 1)$ , where  $k$  is a positive integer. For the  $P_v$ -factorizations of  $K_{m,n}$ , Ushio<sup>[1]</sup> and Wang<sup>[10]</sup> gave a necessary and sufficient condition for their existence when  $v$  is an even number (Theorem 4.9 in ref. [1]).

**Theorem 1**<sup>[1,10]</sup>. Let  $k, m, n$  be positive integers. There exists a  $P_{2k}$ -factorization of  $K_{m,n}$  if and only if  $m = n \equiv 0 \pmod{k(2k-1)}$ .

When  $v$  is an odd number, the existence of the  $P_v$ -factorization of  $K_{m,n}$  seems to be much less tractable. In ref. [1], Ushio proposed the following conjecture (Conjecture 5.3 in ref. [1]).

**Ushio Conjecture**<sup>[1]</sup>. Let  $m, n$  be positive integers and  $k$  be odd. Then  $K_{m,n}$  has a  $P_v$ -factorization if and only if: (1)  $(v-1)m \leq (v+1)n$ , (2)  $(v-1)n \leq (v+1)m$ , (3)  $m+n \equiv 0 \pmod{v}$ , (4)  $vmn/[(v-1)(m+n)]$  is an integer.

Although a lot of work has been done on Ushio Conjecture, up to now, the only known result is that it is true for  $v = 3$ . Very recently, we have done some research in  $P_v$ -factorizations of  $K_{m,n}$ <sup>[11,12]</sup>. In this paper we will show that Ushio Conjecture is true when  $v = 4k - 1$ . That is, we shall prove

**Theorem 2.** Let  $k, m, n$  be positive integers. There exists a  $P_{4k-1}$ -factorization of  $K_{m,n}$  if and only if (1)  $(2k-1)m \leq 2kn$ , (2)  $(2k-1)n \leq 2km$ , (3)  $m+n \equiv 0 \pmod{4k-1}$ , (4)  $(4k-1)mn/[2(2k-1)(m+n)]$  is an integer.

## 1 Proof of necessity

We first give the proof of necessity of Theorem 2.

**Theorem 3.** Let  $k, m, n$  be positive integers. If  $K_{m,n}$  has a  $P_{4k-1}$ -factorization, then (1)  $(2k-1)m \leq 2kn$ , (2)  $(2k-1)n \leq 2km$ , (3)  $m+n \equiv 0 \pmod{4k-1}$ , (4)  $(4k-1)mn/[2(2k-1)(m+n)]$  is an integer.

**Proof.** Let  $r$  be the number of  $P_{4k-1}$ -factors in the factorization,  $e$  be the number of copies  $P_{4k-1}$  in any factor. Using simple computation, we have

$$e = \frac{m+n}{4k-1}, r = \frac{(4k-1)mn}{2(2k-1)(m+n)}.$$

It is easy to see that the conditions (3) and (4) are necessary. Let  $a$  and  $b$  be the number of copies of  $P_{4k-1}$  with its endpoints in  $Y$  and  $X$  in a particular  $P_{4k-1}$ -factor, respectively. Then  $(2k-1)a + 2kb = m$ ,  $2ka + (2k-1)b = n$ . Hence

$$a = \frac{2kn - (2k-1)m}{4k-1}, b = \frac{2km - (2k-1)n}{4k-1}.$$

Conditions (1) and (2) are, therefore, necessary. This proves the theorem.

## 2 Proof of sufficiency

The proof of sufficiency consists of the following lemmas. For any two integers  $x$

and  $y$ , we use  $gcd(x, y)$  to denote the greatest common divisor of  $x$  and  $y$ . The following lemma is obvious.

**Lemma 1.** Let  $u, v, x$  and  $y$  be positive integers. If  $gcd(ux, vy) = 1$ , then  $gcd(uv, ux + vy) = 1$ .

We prove the following result which is used later in this paper.

**Lemma 2.** If  $K_{m,n}$  has a  $P_v$ -factorization, then  $K_{sm,sn}$  has a  $P_v$ -factorization for any positive integer  $s$ .

**Proof.** Let  $\{F_i : 1 \leq i \leq s\}$  be a 1-factorization of  $K_{s,s}$  (see ref. [2]). For each  $i \in \{1, 2, \dots, s\}$ , replace every edge of  $F_i$  by a  $K_{m,n}$  to get a factor  $G_i$  of  $K_{sm,sn}$  such that the graph  $G_i$  are pairwise edge-disjoint and their union is  $K_{sm,sn}$ . Since  $K_{m,n}$  has a  $P_v$ -factorization, it is clear that the graph  $G_i$ , too, has a  $P_v$ -factorization. Consequently,  $K_{sm,sn}$  has a  $P_v$ -factorization.

Lemma 2 implies that there are three cases to consider.

(1)  $(2k - 1)m = 2kn$ .

In this case, let

$$F_j = \{x_{i+2j-1}y_i, x_{i+2j}y_i : 1 \leq i \leq 2k - 1\}, 1 \leq j \leq k,$$

it is easy to see that it is a  $P_{4k-1}$ -factor of  $K_{2k,2k-1}$  and then  $\cup_{1 \leq j \leq k} F_j$  is a  $P_{4k-1}$ -factorization of  $K_{2k,2k-1}$ .  $K_{m,n}$  has a  $P_{4k-1}$ -factorization from Lemma 2.

(2)  $2km = (2k - 1)n$ .

Obviously,  $K_{m,n}$  has a  $P_{4k-1}$ -factorization.

(3)  $2km > (2k - 1)n$  and  $2kn > (2k - 1)m$ .

In this case, let

$$a = \frac{2kn - (2k - 1)m}{4k - 1}, \quad b = \frac{2km - (2k - 1)n}{4k - 1},$$

$$e = \frac{m + n}{4k - 1}, \quad r = \frac{(4k - 1)mn}{(4k - 2)(m + n)}.$$

Then from conditions (1)–(4) in Theorem 2,  $a, b, e, r$  are integers and  $0 < a < m, 0 < b < n$ . We have  $(2k - 1)a + 2kb = m, (2k - 1)b + 2ka = n$ . Hence

$$r = k(a + b) + \frac{ab}{2(2k - 1)(a + b)}.$$

Let

$$z = \frac{ab}{2(2k - 1)(a + b)},$$

which is a positive integer. And let  $gcd((2k - 1)a, 2kb) = d, (2k - 1)a = dp, 2kb = dq$ , where  $gcd(p, q) = 1$ . Then

$$z = \frac{dpq}{2(2k - 1)[2kp + (2k - 1)q]}.$$

These equalities imply the following equalities:

$$\begin{aligned}d &= \frac{2(2k-1)[2kp+(2k-1)q]z}{pq}, & r &= \frac{(p+q)[4k^2p+(2k-1)^2q]z}{pq}, \\m &= \frac{2(2k-1)(p+q)[2kp+(2k-1)q]z}{pq}, \\n &= \frac{[4k^2p+(2k-1)^2q][2kp+(2k-1)q]z}{kpq}, \\a &= \frac{2p[2kp+(2k-1)q]z}{pq}, & b &= \frac{(2k-1)q[2kp+(2k-1)q]z}{kpq}.\end{aligned}$$

Let  $2k-1 = p_1^{k_1} p_2^{k_2} \cdots p_\gamma^{k_\gamma}$ , where  $p_1, p_2, \dots, p_\gamma$  are distinct prime numbers,  $k_1, k_2, \dots, k_\gamma$  are positive integers, and  $2k = q_1^{h_1} q_2^{h_2} \cdots q_\omega^{h_\omega}$ , where  $q_1, q_2, \dots, q_\omega$  are distinct prime numbers,  $h_1, h_2, \dots, h_\omega$  are positive integers. Now we can establish the following lemma.

**Lemma 3.** If  $\gcd(p, (2k-1)^2) = p_1^{i_1} p_2^{i_2} \cdots p_\alpha^{i_\alpha} p_{\alpha+1}^{2k_{\alpha+1}-i_{\alpha+1}} p_{\alpha+2}^{2k_{\alpha+2}-i_{\alpha+2}} \cdots p_\beta^{2k_\beta-i_\beta} p_{\beta+1}^{2k_{\beta+1}} p_{\beta+2}^{2k_{\beta+2}} \cdots p_\gamma^{2k_\gamma}$ , where  $1 \leq \alpha \leq \beta \leq \gamma$ ,  $0 \leq i_j \leq k_j$  (when  $1 \leq j \leq \alpha$ ) or  $0 < i_j < k_j$  (when  $\alpha+1 \leq j \leq \beta$ );  $\gcd(q, 4k^2) = q_1^{j_1} q_2^{j_2} \cdots q_\mu^{j_\mu} q_{\mu+1}^{2h_{\mu+1}-j_{\mu+1}} q_{\mu+2}^{2h_{\mu+2}-j_{\mu+2}} \cdots q_\nu^{2h_\nu-j_\nu} q_{\nu+1}^{2h_{\nu+1}} q_{\nu+2}^{2h_{\nu+2}} \cdots q_\omega^{2h_\omega}$ , where  $1 \leq \mu \leq \nu \leq \omega$ ,  $0 \leq j_i \leq h_i$  (when  $1 \leq i \leq \mu$ ) or  $0 < j_i < h_i$  (when  $\mu+1 \leq i \leq \nu$ ). Let

$$\begin{aligned}s &= p_1^{i_1} p_2^{i_2} \cdots p_\alpha^{i_\alpha}, & t &= p_1^{k_1-i_1} p_2^{k_2-i_2} \cdots p_\alpha^{k_\alpha-i_\alpha}, \\u &= p_{\alpha+1}^{i_{\alpha+1}} p_{\alpha+2}^{i_{\alpha+2}} \cdots p_\beta^{i_\beta}, \\v &= p_{\alpha+1}^{k_{\alpha+1}-i_{\alpha+1}} p_{\alpha+2}^{k_{\alpha+2}-i_{\alpha+2}} \cdots p_\beta^{k_\beta-i_\beta}, \\w &= p_{\beta+1}^{k_{\beta+1}} p_{\beta+2}^{k_{\beta+2}} \cdots p_\gamma^{k_\gamma}, \\s' &= q_1^{j_1} q_2^{j_2} \cdots q_\mu^{j_\mu}, & t' &= q_1^{h_1-j_1} q_2^{h_2-j_2} \cdots q_\mu^{h_\mu-j_\mu}, \\u' &= q_{\mu+1}^{j_{\mu+1}} q_{\mu+2}^{j_{\mu+2}} \cdots q_\nu^{j_\nu}, \\v' &= q_{\mu+1}^{h_{\mu+1}-j_{\mu+1}} q_{\mu+2}^{h_{\mu+2}-j_{\mu+2}} \cdots q_\nu^{h_\nu-j_\nu}, \\w' &= q_{\nu+1}^{h_{\nu+1}} q_{\nu+2}^{h_{\nu+2}} \cdots q_\omega^{h_\omega},\end{aligned}$$

and let  $p = suv^2 w^2 p'$ ,  $q = s' u' v'^2 w'^2 q'$ .

(1) If  $t' \equiv 1 \pmod{2}$  and  $v' w' \equiv 1 \pmod{2}$ , then

$$\begin{aligned}m &= 2stut'(suv^2 w^2 p' + s' u' v'^2 w'^2 q')(vwt' p' + tv' w' q') z', \\n &= 2suvwv' w' (s' t'^2 u' p' + st^2 uq')(vwt' p' + tv' w' q') z', \\a &= 2suvw t' p' (vwt' p' + tv' w' q') z', \\b &= 2stuv' w' q' (vwt' p' + tv' w' q') z', \\r &= t' v' w' (suv^2 w^2 p' + s' u' v'^2 w'^2 q')(s' t'^2 u' p' + st^2 uq') z', \\d &= 2stut' (vwt' p' + tv' w' q') z',\end{aligned}$$

for some positive integer  $z'$ .

(2) If  $t' \equiv 0 \pmod{2}$  and  $v'w' \equiv 1 \pmod{2}$ , then

$$\begin{aligned} m &= stut'(suv^2w^2p' + s'u'v'^2w'^2q')(vwt'p' + tv'w'q')z', \\ n &= suv'vw'w'(s't'^2u'p' + st^2uq')(vwt'p' + tv'w'q')z', \\ a &= suvwt'p'(vwt'p' + tv'w'q')z', \\ b &= stuv'w'q'(vwt'p' + tv'w'q')z', \\ r &= t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z'/2, \\ d &= stut'(vwt'p' + tv'w'q')z', \end{aligned}$$

for some positive integer  $z'$ .

(3) If  $t' \equiv 1 \pmod{2}$  and  $v'w' \equiv 0 \pmod{2}$ , then

$$\begin{aligned} m &= stut'(suv^2w^2p' + s'u'v'^2w'^2q')(vwt'p' + tv'w'q')z', \\ n &= suv'vw'w'(s't'^2u'p' + st^2uq')(vwt'p' + tv'w'q')z', \\ a &= suvwt'p'(vwt'p' + tv'w'q')z', \\ b &= stuv'w'q'(vwt'p' + tv'w'q')z', \\ r &= t'v'w'(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z'/2, \\ d &= stut'(vwt'p' + tv'w'q')z', \end{aligned}$$

for some positive integer  $z'$ .

**Proof.** We assume the  $\gcd(p, (2k - 1)^2) = suv^2w^2$ ,  $\gcd(q, 4k^2) = s'u'v'^2w'^2$ ,  $\gcd(p, q) = 1$  and  $p = suv^2w^2p'$ ,  $q = s'u'v'^2w'^2q'$  holds. Then

$$\gcd(suv^2w^2p', s'u'v'^2w'^2q') = \gcd(s't'^2u'p', st^2uq') = 1.$$

Since

$$r = \frac{(suv^2w^2p' + s'u'v'^2w'^2q')(s't'^2u'p' + st^2uq')z}{p'q'}.$$

From Lemma 1, we see that  $\gcd(suv^2w^2p' + s'u'v'^2w'^2q', p'q') = \gcd(s't'^2u'p' + st^2uq', p'q') = 1$ . Hence  $z/(p'q')$  is an integer. Let  $z_1 = z/(p'q')$ , we have

$$d = \frac{2stu(vwt'p' + tv'w'q')z_1}{v'w'}.$$

(1) When  $t' \equiv 1 \pmod{2}$  and  $v'w' \equiv 1 \pmod{2}$

Since  $\gcd(2, v'w') = \gcd(stu, v'w') = \gcd(vwt'p' + tv'w'q', v'w') = 1$ , therefore  $z_1/(v'w')$  is an integer. Let  $z_2 = z_1/(v'w')$ , we have

$$b = \frac{2stuv'w'q'(vwt'p' + tv'w'q')z_2}{t'}.$$

Since  $\gcd(2, t') = \gcd(st'uv'w'q', t') = \gcd(vwt'p' + tv'w'q', t') = 1$ , therefore  $z_2/t'$  is an integer. Let  $z' = z_2/t'$ . Then the equalities in (1) hold.

(2) When  $t' \equiv 0 \pmod{2}$  and  $v'w' \equiv 1 \pmod{2}$

Since  $\gcd(2, v'w') = \gcd(stu, v'w') = \gcd(vwt'p' + tv'w'q', v'w') = 1$ , therefore

$z_1/(v'w')$  is an integer. Let  $z_2 = z_1/(v'w')$ , we have

$$b = \frac{2stuv'w'q'(vwt'p' + tv'w'q')z_2}{t'}.$$

Since  $\gcd(2, t') = 2$ ,  $\gcd(st'uv'w'q', t') = \gcd(vwt'p' + tv'w'q', t') = 1$ , therefore  $2z_2/t'$  is an integer. Let  $z' = 2z_2/t'$ . Then the equalities in (2) hold.

(3) When  $t' \equiv 1 \pmod{2}$  and  $v'w' \equiv 0 \pmod{2}$

Since  $\gcd(2, v'w') = 2$ ,  $\gcd(stu, v'w') = \gcd(vwt'p' + tv'w'q', v'w') = 1$ , therefore  $2z_1/(v'w')$  is an integer. Let  $z_2 = 2z_1/(v'w')$ , we have

$$b = \frac{stuv'w'q'(vwt'p' + tv'w'q')z_2}{t'}.$$

Since  $\gcd(st'uv'w'q', t') = \gcd(vwt'p' + tv'w'q', t') = 1$ , therefore  $z_2/t'$  is an integer. Let  $z' = z_2/t'$ . Then the equalities in (3) hold.

This proves the lemma.

We are now in position to prove Theorem 2. For our main result, we only need the following direct constructions.

**Lemma 4.** For any positive integers  $s, t, u, v, w, s', t', u', v', w', p$  and  $q$ , let

$$\begin{aligned} m &= 2stut'(suv^2w^2p + s'u'v'^2w'^2q)(vwt'p + tv'w'q), \\ n &= 2suvvw'w'(s't'^2u'p + st^2uq)(vwt'p + tv'w'q). \end{aligned}$$

Then  $K_{m,n}$  has a  $P_{stuvw+s't'u'v'w'}$ -factorization if  $stuvw + 1 = s't'u'v'w'$ .

**Proof.** Let  $a = 2suvwt'p(vwt'p + tv'w'q)$ ,  $b = 2stuv'w'q(vwt'p + tv'w'q)$ ,  $r = t'v'w'(suv^2w^2p + s'u'v'^2w'^2q)(s't'^2u'p + st^2uq)$ ,  $r_1 = t'(suv^2w^2p + s'u'v'^2w'^2q)$  and  $r_2 = v'w'(s't'^2u'p + st^2uq)$ . Let  $X$  and  $Y$  be the two partite sets of  $K_{m,n}$  and set

$$\begin{aligned} X &= \{x_{i,j} : 1 \leq i \leq r_1; 1 \leq j \leq 2stu(vwt'p + tv'w'q)\}, \\ Y &= \{y_{i,j} : 1 \leq i \leq r_2; 1 \leq j \leq 2suvw(vwt'p + tv'w'q)\}. \end{aligned}$$

We shall construct a  $P_{stuvw+s't'u'v'w'}$ -factorization of  $K_{m,n}$ . We remark in advance that the additions in the first subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are taken modulo  $r_1$  and  $r_2$  in  $\{1, 2, \dots, r_1\}$  and  $\{1, 2, \dots, r_2\}$ , respectively, and the additions in the second subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are taken modulo  $2stu(vwt'p + tv'w'q)$  and  $2suvw(vwt'p + tv'w'q)$  in  $\{1, 2, \dots, 2stu(vwt'p + tv'w'q)\}$  and  $\{1, 2, \dots, 2suvw(vwt'p + tv'w'q)\}$ , respectively.

For each  $i, x, y, z$  and  $x'$ ,  $1 \leq i \leq t'p$ ,  $1 \leq x \leq vw$ ,  $1 \leq y \leq suvw$ ,  $1 \leq z \leq t$  and  $0 \leq x' \leq 1$ , let  $f(i, x, y) = suv^2w^2(i-1) + suvw(x-1) + y$ ,  $g(i, y, z, x') = s't'u'v'w'(i-1) + suvw(z-1) + y + x'$  and  $h(i, x, y, x') = 2suvw(i-1) + 2su(vwt'p + tv'w'q)(x-1) + 2y + x' - 1$ , and set

$$\begin{aligned} E_i &= \{x_{f(i,x,y),j+2su(vwt'p+tv'w'q)(z-1)}y_{g(i,y,z,x'),j+h(i,x,y,x')} : \\ &1 \leq j \leq 2su(vwt'p + tv'w'q), 1 \leq x \leq vw, \\ &1 \leq y \leq suvw, 1 \leq z \leq t, 0 \leq x' \leq 1\}. \end{aligned}$$

For each  $i, x, y, z$  and  $x', 1 \leq i \leq v'w'q, 1 \leq x \leq stu, 1 \leq y \leq vw, 1 \leq z \leq t$  and  $0 \leq x' \leq 1$ , let  $\varphi(i, x, y, x') = suv^2w^2t'p + s't'u'v'w'(i - 1) + vw(x - 1) + y + x', \psi(i, x, z) = s't'^2u'v'w'p + st^2u(i - 1) + stu(z - 1) + x$  and  $\phi(i, x, y, x') = 2suvwt'p + 2x + 2stu(i - 1) + 2su(vwt'p + tv'w'q)(y - 1) + x' - 1$ , and let

$$E_{t'p+i} = \{x_{\varphi(i,x,y,x'),j+2su(vwt'p+tv'w'q)(z-1)}y_{\psi(i,x,z),j+\phi(i,x,y,x')} : \\ 1 \leq j \leq 2su(vwt'p + tv'w'q), 1 \leq x \leq stu, \\ 1 \leq y \leq vw, 1 \leq z \leq t, 0 \leq x' \leq 1\}.$$

Notice that  $stuvw + 1 = s't'u'v'w'$ . Let  $F = \cup_{1 \leq i \leq t'p+v'w'q} E_i$ , then it is easy to see that the graph  $F$  is a  $P_{stuvw+s't'u'v'w'}$ -factor of  $K_{m,n}$ . Define a bijection  $\sigma$  from  $X \cup Y$  onto  $X \cup Y$  in such a way that  $\sigma(x_{i,j}) = x_{i+1,j}, \sigma(y_{i,j}) = y_{i+1,j}$ . For each  $i \in \{1, 2, \dots, r_1\}$  and each  $j \in \{1, 2, \dots, r_2\}$ , let

$$F_{i,j} = \{\sigma^i(x)\sigma^j(y) \mid x \in X, y \in Y, xy \in F\}.$$

It is easy to show that the graphs  $F_{i,j} (1 \leq i \leq r_1, 1 \leq j \leq r_2)$  are  $P_{stuvw+s't'u'v'w'}$ -factors of  $K_{m,n}$  and their union is  $K_{m,n}$ . Thus  $\{F_{i,j} \mid 1 \leq i \leq r_1, 1 \leq j \leq r_2\}$  is a  $P_{stuvw+s't'u'v'w'}$ -factorization of  $K_{m,n}$ .

This proves the lemma.

In the following constructions,  $\lceil x \rceil$  denotes the least integer greater than or equal to  $x$  and  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ .

**Lemma 5.** For any positive integers  $s, t, u, v, w, s', t', u', v', w', p$  and  $q, t' \equiv 0 \pmod{2}$ , let

$$m = stut'(suv^2w^2p + s'u'v'^2w'^2q)(vwt'p + tv'w'q), \\ n = suvwv'w'(s't'^2u'p + st^2uq)(vwt'p + tv'w'q),$$

Then  $K_{m,n}$  has a  $P_{stuvw+s't'u'v'w'}$ -factorization if  $stuvw + 1 = s't'u'v'w'$ .

**Proof.** Let  $a = suvwt'p(vwt'p + tv'w'q), b = stuw'w'q(vwt'p + tv'w'q), r = t'v'w'(suv^2w^2p + s'u'v'^2w'^2q)(s't'^2u'p + st^2uq)/2, r_1 = t'(suv^2w^2p + s'u'v'^2w'^2q)/2$  and  $r_2 = v'w'(s't'^2u'p + st^2uq)$ . Let  $X$  and  $Y$  be the two partite sets of  $K_{m,n}$  and set

$$X = \{x_{i,j} : 1 \leq i \leq r_1; 1 \leq j \leq 2stu(vwt'p + tv'w'q)\}, \\ Y = \{y_{i,j} : 1 \leq i \leq r_2; 1 \leq j \leq suvw(vwt'p + tv'w'q)\}.$$

We shall construct a  $P_{stuvw+s't'u'v'w'}$ -factorization of  $K_{m,n}$ . We remark in advance that the additions in the first subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are the same as those of Lemma 4, and the additions in the second subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are taken modulo  $2stu(vwt'p + tv'w'q)$  and  $suvw(vwt'p + tv'w'q)$  in  $\{1, 2, \dots, 2stu(vwt'p + tv'w'q)\}$  and  $\{1, 2, \dots, suvw(vwt'p + tv'w'q)\}$ , respectively.

For each  $i, x, y, z, x'$  and  $y', 1 \leq i \leq pt'/2, 1 \leq x \leq vw, 1 \leq y \leq suvw, 1 \leq z \leq t$  and  $0 \leq x', y' \leq 1$ , let  $f(i, x, y) = suv^2w^2(i - 1) + suvw(x - 1) + y, g(i, y, z, x', y') = 2s't'u'v'w'(i - 1) + s't'u'v'w'x' + suvw(z - 1) + y + y'$  and

$h(i, x, y, y') = 2suvw(i - 1) + su(vwt'p + tv'w'q)(x - 1) + 2y + y' - 1$ , and set

$$E_i = \{x_{f(i,x,y),j+su(vwt'p+tv'w'q)(z-1)+stu(vwt'p+tv'w'q)x'} y_{g(i,y,z,x',y'),j+h(i,x,y,y')} : \\ 1 \leq j \leq su(vwt'p + tv'w'q), 1 \leq x \leq vw, \\ 1 \leq y \leq suvw, 1 \leq z \leq t, 0 \leq x', y' \leq 1\}.$$

For each  $i, x, y, z$  and  $x', 1 \leq i \leq v'w'q, 1 \leq x \leq stu, 1 \leq y \leq vw, 1 \leq z \leq t$  and  $0 \leq x' \leq 1$ , let  $\varphi(i, x, y) = suv^2w^2t'p/2 + s't'u'v'w'(i - 1)/2 + \lceil [vw(x - 1) + y + x']/2 \rceil$ ,  $\psi(x, y, z, x') = su(vwt'p + tv'w'q)(z - 1) + stu(vwt'p + tv'w'q)[1 - (\lceil [vw(x - 1) + y]/2 \rceil - \lfloor [vw(x - 1) + y]/2 \rfloor)](1 - x') + stu(vwt'p + tv'w'q)(\lceil [vw(x - 1) + y]/2 \rceil - \lfloor [vw(x - 1) + y]/2 \rfloor)x'$  and  $\phi(i, x, y) = suvwt'p + x + stu(i - 1) + su(vwt'p + tv'w'q)(y - 1)$ , and let

$$E_{t'p/2+i} = \{x_{\varphi(i,x,y),j+\psi(x,y,z,x')} y_{s't'^2u'v'w'p+st^2u(i-1)+stu(z-1)+x,j+\phi(i,x,y)} : \\ 1 \leq j \leq su(vwt'p + tv'w'q), 1 \leq x \leq stu, \\ 1 \leq y \leq vw, 1 \leq z \leq t, 0 \leq x' \leq 1\}.$$

The remaining proof is similar as that of Lemma 4.

**Lemma 6.** For any positive integers  $s, t, u, v, w, s', t', u', v', w', p$  and  $q, v'w' \equiv 0 \pmod{2}$ , let

$$m = stut'(suv^2w^2p + s'u'v'^2w'^2q)(vwt'p + tv'w'q), \\ n = suvwv'w'(s't'^2u'p + st^2uq)(vwt'p + tv'w'q).$$

Then  $K_{m,n}$  has a  $P_{stuvw+s't'u'v'w'}$ -factorization if  $stuvw + 1 = s't'u'v'w'$ .

**Proof.** Let  $a = suvwt'p(vwt'p + tv'w'q)$ ,  $b = stw'w'q(vwt'p + tv'w'q)$ ,  $r = t'v'w'(suv^2w^2p + s'u'v'^2w'^2q)(s't'^2u'p + st^2uq)/2$ ,  $r_1 = t'(suv^2w^2p + s'u'v'^2w'^2q)$  and  $r_2 = v'w'(s't'^2u'p + st^2uq)/2$ . Let  $X$  and  $Y$  be the two partite sets of  $K_{m,n}$  and set

$$X = \{x_{i,j} : 1 \leq i \leq r_1; 1 \leq j \leq stu(vwt'p + tv'w'q)\}, \\ Y = \{y_{i,j} : 1 \leq i \leq r_2; 1 \leq j \leq 2suvw(vwt'p + tv'w'q)\}.$$

We shall construct a  $P_{stuvw+s't'u'v'w'}$ -factorization of  $K_{m,n}$ . We remark in advance that the additions in the first subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are the same as those of Lemma 4, and the additions in the second subscripts of  $x_{i,j}$ 's and  $y_{i,j}$ 's are taken modulo  $stu(vwt'p + tv'w'q)$  and  $2suvw(vwt'p + tv'w'q)$  in  $\{1, 2, \dots, stu(vwt'p + tv'w'q)\}$  and  $\{1, 2, \dots, 2suvw(vwt'p + tv'w'q)\}$ , respectively.

For each  $i, x, y, z$  and  $x', 1 \leq i \leq t'p, 1 \leq x \leq vw, 1 \leq y \leq suvw, 1 \leq z \leq t$  and  $0 \leq x' \leq 1$ , let  $f(i, x, y) = suv^2w^2(i - 1) + suvw(x - 1) + y$ ,  $g(i, y, z, x') = s't'u'v'w'(i - 1)/2 + \lceil (t(y - 1) + z + x')/2 \rceil$  and  $h(i, x, y, z, x') = suvw(i - 1) + su(vwt'p + tv'w'q)(x - 1) + y + suvw(vwt'p + tv'w'q)[1 - (\lceil (t(y - 1) + z)/2 \rceil - \lfloor (t(y - 1) + z)/2 \rfloor)](1 - x') + suvw(vwt'p + tv'w'q)(\lceil (t(y - 1) + z)/2 \rceil - \lfloor (t(y - 1) + z)/2 \rfloor)x'$ ,

and let

$$E_i = \{x_{f(i,x,y),j+su(vwt'p+tv'w'q)(z-1)}y_{g(i,y,z,x'),j+h(i,x,y,z,x')} : \\ 1 \leq j \leq su(vwt'p + tv'w'q), 1 \leq x \leq vw, \\ 1 \leq y \leq suvw, 1 \leq z \leq t, 0 \leq x' \leq 1\}.$$

For each  $i, x, y, z, x'$  and  $y', 1 \leq i \leq v'w'q/2, 1 \leq x \leq stu, 1 \leq y \leq vw, 1 \leq z \leq t$  and  $0 \leq x', y' \leq 1$ , let  $\varphi(i, x, y, x', y') = sv^2w^2t'p + s't'u'v'w'y' + 2s't'u'v'w'(i - 1) + vw(x - 1) + y + x', \psi(i, x, z) = s't'^2u'v'w'p/2 + st^2u(i - 1) + stu(z - 1) + x$  and  $\phi(i, x, y, x', y') = suvwt'p + x + stux' + 2stu(i - 1) + su(vwt'p + tv'w'q)(y - 1) + suvw(vwt'p + tv'w'q)y'$ , and let

$$E_{t'p+i} = \{x_{\varphi(i,x,y,x',y'),j+su(vwt'p+tv'w'q)(z-1)}y_{\psi(i,x,z),j+\phi(i,x,y,x',y')} : \\ 1 \leq j \leq su(vwt'p + tv'w'q), 1 \leq x \leq stu, \\ 1 \leq y \leq vw, 1 \leq z \leq t, 0 \leq x', y' \leq 1\}.$$

The remaining proof is similar as that of Lemma 4.

Applying Lemmas 2—6, we see that for the parameters  $k, m$  and  $n$  satisfying conditions (1)—(4) in Theorem 2,  $K_{m,n}$  has a  $P_{4k-1}$ -factorization.

**Theorem 4.** Let  $k, m, n$  be positive integers. If (1)  $(2k - 1)m \leq 2kn$ , (2)  $(2k - 1)n \leq 2km$ , (3)  $m + n \equiv 0 \pmod{4k - 1}$ , (4)  $(4k - 1)mn/[2(2k - 1)(m + n)]$  is an integer, then  $K_{m,n}$  has a  $P_{4k-1}$ -factorization.

Combining Theorem 3 and Theorem 4 we obtain Theorem 2, that is, we have proved: Ushio Conjecture on  $P_v$ -factorization of  $K_{m,n}$  is true when  $v = 4k - 1$ .

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**References**

1. Ushio, K.,  $G$ -designs and related designs, *Discrete Math.*, 1993, 116: 299—311.
2. Bondy, J. A., Murty, U. S. R., *Graph Theory with Applications*, London Basingstoke: Macmillan Press, 1976.
3. Yamamoto, S., Ikeda, H., Shige-eda, S. et al., Design of a new balanced file organization scheme with the least redundancy, *Information and Control*, 1975, 28: 156—175.
4. Yamamoto, S., Tazawa, S., Ushio, K. et al., Design of a generalized balanced multiple-valued file organization scheme with the least redundancy, *ACM Trans. Database Systems*, 1979, 4: 518—530.
5. Ushio, K.,  $P_3$ -factorization of complete bipartite graphs, *Discrete Math.*, 1988, 72: 361—366.
6. Martin, N., Complete bipartite factorisations by complete bipartite graphs, *Discrete Math.*, 1997, 167/168: 461—480.
7. Du, B. L.,  $K_{1,p^2}$ -factorization of complete bipartite graphs, *Discrete Math.*, 1998, 187: 273—279.
8. Du, B. L., Wang, J.,  $K_{1,k}$ -factorizations of complete bipartite graphs, *Discrete Math.*, 2002, 259: 301—306.
9. Du, B. L., Wang, J.,  $K_{p,q}$ -factorization of complete bipartite graphs, *Science in China, Ser. A*, 2004, 47(3): 473—479.
10. Wang, H.,  $P_{2k}$ -factorization of a complete bipartite graph, *Discrete Math.*, 1993, 120: 307—308.
11. Du, B. L.,  $P_{2k}$ -factorization of complete bipartite multigraphs, *Austral. J. Combin*, 2000, 21: 197—199.
12. Wang, J., Du, B. L.,  $\hat{P}_3$ -factorization of complete bipartite multigraphs and symmetric complete bipartite multigraphs, *Utilitas Math.*, 2003, 63: 213—228.